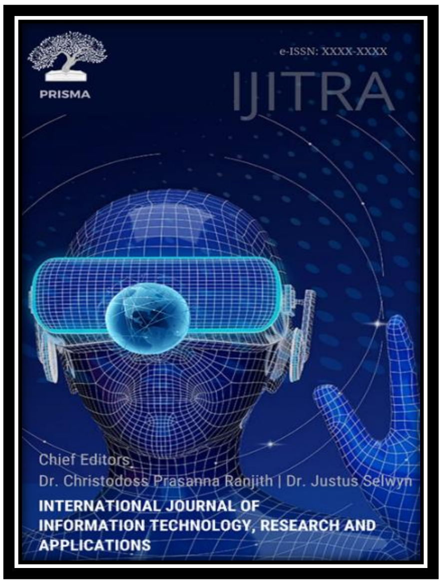

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SOLVING FUZZY FRACTIONAL HEAT EQUATION USING HOMOTOPY PERTURBATION SUMUDU TRANSFORM METHOD

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ABSTRACT

This paper extends the Homotopy Perturbation Sumudu Transform Method (HPSTM) to solve fuzzy fractional heat equations. To illustrate the reliability of the method, some examples are presented. The results reveal that HPSTM is a highly effective scheme for obtaining approximate analytical solutions of fuzzy fractional heat equations. Figures and numerical examples demonstrate the expertise of the suggested approach. This method is applied for both linear and nonlinear ordinary and partial FFDEs. The proposed approach is rapid, exact, and simple to apply and produce excellent outcomes.

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1. INTRODUCTION

Fractional calculus is expanded upon in ordinary calculus. This involves computing a function's derivative in any order. The memory and heredity characteristics of numerous substances and processes in porous media, electrical circuits, control, biology, electromagnetic processes, biomechanics, and electrochemistry have been documented using fractional differential operators [1]-[5]. Over the past few decades, fractional calculus and its applications have gained popularity, mostly because it has proven to be a helpful tool for modeling a number of complex processes in a wide range of seemingly unrelated fields of science and engineering[6]-[9]. Measurement uncertainty is represented by a fuzzy number. Since its invention by Lotfi Zadeh in 1965, fuzzy sets have found utility in a variety of contexts [10]-[12].

The solution for fractional differential equations in uncertain settings was first introduced by Agarval in 2010. Amr M. S. Mahdy et al. [14] implemented the Homotopy Perturbation Sumudu Transform Method for Solving Klein-Gordon Equation There have been many implementations in the area of Fuzzy Fractional Differential Equations (FFDEs) and Fuzzy Fractional Partial Differential Equations (FFPDEs) to find precise and numerical solutions. Consider the general FFPDE with fuzzy initial conditions of the form:

$$D_t^{\alpha}\tilde{w}(x,t)+L\tilde{w}(x,t)+N\tilde{w}(x,t)=g(x,t)$$

with fuzzy initial conditions $\tilde{w}(\mathrm{x},0)=\tilde{g}(\mathrm{x})$. Where $D_t^{\alpha}=\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ represents the Caputo fractional derivative of $\tilde{w}(x,t)$ and $\tilde{w}(x,t)$ is the fuzzy function. The general nonlinear and linear differential operators are denoted by N and L. The source term is $\mathrm{g}(\mathrm{x},t)$.

The focus of this work is to calculate the analytical solution for the fuzzy fractional heat equation in the form

$$\frac{\partial^{\alpha}\tilde{w}}{\partial t^{\alpha}}=\frac{x^2}{2}\frac{\partial^2\tilde{w}}{\partial t^2},\qquad 0<x<1,\qquad t>0,\qquad 0<\alpha<1$$

with initial condition $\tilde{w}(x,0)=\tilde{f}(x)$.

To solve FFPDEs, various techniques have been devised [15]. For the nonlinear problem, the exact solution is difficult to obtain because of its complexity. So here the Sumudu transform is combined with the homotopy perturbation method called the Homotopy Perturbation Sumudu Transform Method (HPSTM) has been proposed. The considered technique is unique in that it uses a simple method to assess the result and is based on the perturbation technique, which allows quick convergence to the exact solution of the problem. When compared to traditional methods, the proposed method can reduce the volume of computing effort while retaining high numerical accuracy; the size reduction equates to an improvement in the approach's performance.

This paper is organized as follows. Section 2, gives some basic tools that involve fractional calculus and fuzzy numbers. Section 3, describes the homotopy perturbation Sumudu transform method for fuzzy fractional differential equations. Section 4, contains examples to show the efficiency of using HPSTM to solve fuzzy fractional heat equations. Finally, Section 4 offers the conclusion of this paper.

2. BASIC TOOLS

The essential definitions of fractional calculus, fuzzy numbers, and Sumudu transform are provided in this section.

2.1 Fractional Calculus

Fractional calculus deals with generalizations of integer order derivatives integrals to arbitrary order. This section presents the basic definitions and properties which will be used in the subsequent section [13].

If $f(x) \in C[a, b]$ and $\alpha > 0$, then

$${}_aI_x^{\alpha} f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt,$$

$${}_aD_x^{\alpha} f(x):=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x \frac{f(t)}{(x-t)^{-n+\alpha+1}}dt,$$

$${}_x^CD_b^{\alpha} f(x)=\frac{1}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1}\left(-\frac{d}{dt}\right)^n f(t)dt,$$

$${}_aI_t^{\alpha}\,{}_aD_t^{\alpha}x(t)=x(t)-\sum_{j=1}^{n}\frac{{}_aD_t^{\alpha-j}}{\Gamma(\alpha+1-j)}(t-a)^{\alpha-j}$$

are called the left-sided Riemann-Liouville (RL), fractional integral Riemann-Liouville, the fractional derivative of order α, and left-sided Caputo fractional derivatives, respectively.

Definition 2.1

The Mittag-Leffler function is defined as follows,

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(n\alpha+1)}, \qquad \alpha \in C, \qquad \mathrm{Re}(\alpha)>0$$

A further generalization of the above equation is defined in the form

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(n\alpha+\beta)}, \qquad \alpha,\beta \in C, \qquad \mathrm{Re}(\alpha)>0, \qquad \mathrm{Re}(\beta)>0.$$

2.2 Fuzzy Number [13]

Definition 2.2 (Triangular fuzzy number)

It's a three-pointed fuzzy number represented by $N=(\theta_1,\theta_2,\theta_3)$. N's membership function is as follows:

$$\mu_N(\chi)=\begin{cases} 0, & \chi<\theta_1 \\ \dfrac{\chi-\theta_1}{\theta_2-\theta_1}, & \theta_1\le\chi\le\theta_2 \\ \dfrac{\theta_3-\chi}{\theta_3-\theta_2}, & \theta_2\le\chi\le\theta_3 \\ 0, & \chi>\theta_3 \end{cases}$$

2.3 Sumudu Transform

Definition 2.3

The Sumudu transform is defined over the set of functions:

$$A=\left\{ f(t)\,|\,\exists,\tau_1,\tau_2>0, |\,f(t)\,|<Me^{\frac{|t|}{t_j}} \qquad if \qquad t\in(-1)^j\times[0,\infty) \right\}$$

By the following formula:

$$\overline{f}(u)=S[f(t)]=\int_{0}^{\infty} f(ut)e^{-t}dt, \qquad u\in(\tau_1,\tau_2)$$

Some special properties of the Sumudu transform are as follows:

1. $S[1]=1;$
2. $S\left[\dfrac{t^{m}}{\Gamma(m+1)}=u^{m}; \qquad m>0\right]$

Definition 2.4

The sumudu transform of the Caputo fractional derivative is defined as follows:

$$S[D_t^{\alpha} f(x,t)]=u^{-\varepsilon}S[f(x,t)]-\sum_{k=0}^{m-1}u^{-\alpha+k}f^{(k)}(0^{+}), \qquad m-1<\alpha\le m.$$

3. HOMOTOPY PERTURBATION SUMUDU TRANSFORM METHOD FOR FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, it is explained the results of research and at the same time is given the comprehensive discussion.

To illustrate the basic idea of this method, consider a general fuzzy fractional partial differential equation with the initial condition of the form:

$$D_t^{\alpha}\tilde{w}(x,t)+L\tilde{w}(x,t)+N\tilde{w}(x,t)=g(x,t) \tag{3.1}$$

with $m-1<\alpha\leq m$, and subject to the initial condition

$$\frac{\partial^s\tilde{w}(x,t)}{\partial t^s}=\tilde{w}^s(x,0), \qquad s=0,1,2,....,m-1.$$

where $D_t^{\alpha}=\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivative of the function $\tilde{w}(x,t)=[\underline{w}(x,t),\overline{w}(x,t)]$,

$g(x, t)$ is the source term, L is the linear operator and N is the general nonlinear differential operator. The parametric form of (3.1) is,

$$D_t^{\alpha}\underline{w}(x,t)+L\underline{w}(x,t)+N\underline{w}(x,t)=g(x,t) \tag{3.2}$$

$$D_t^{\alpha}\overline{w}(x,t)+L\overline{w}(x,t)+N\overline{w}(x,t)=g(x,t) \tag{3.3}$$

Applying the Sumudu transform (denoted by this paper by S) on both sides of equation (3.2),

$$S[D_t^{\alpha}\underline{w}(x,t)]+S[L\underline{w}(x,t)]+S[N\underline{w}(x,t)]=S[g(x,t)]$$

Using the differentiation property of the Sumudu transform and the initial conditions in equation (3.1),

$$S[\underline{w}(x,t)]=\sum_{k=0}^{m-1}u^{-\alpha+k}\,\underline{w}^{(k)}(x,0)+u^{\alpha}S[g(x,t)]-u^{\alpha}S[L\underline{w}(x,t)]+N\underline{w}(x,t) \tag{3.4}$$

Operating with the Sumudu inverse on both sides of equation (3.4) gives

$$\underline{w}(x,t)=G(x,t)-S^{-1}[u^{\alpha}S[L\underline{w}(x,t)+N\underline{w}(x,t)-g(x,t)]] \tag{3.5}$$

where $G(x, t)$ represent the prescribed initial conditions. Now apply the HPM.

$$\underline{w}(x,t)=\sum_{n=0}^{\infty}p^n\,\underline{w}_n(x,t) \tag{3.6}$$

and the nonlinear term can be decomposed as

$$N\underline{w}(x,t)=\sum_{n=0}^{\infty}p^nA_n \tag{3.7}$$

for some Adomian's polynomials A_n that are given by [16],

$$A_n=\frac{1}{n!}\frac{d^n}{dp^n}\left[N\left(\sum_{i=0}^{\infty}p^i\,\underline{w}_i\right)\right]_{p=0}, n=0,1,2,...$$

Substituting equation (3.6) and (3.7) in equation (3.5),

$$\sum_{n=0}^{\infty} p^n \underline{w}_n(x,t) = G(x,t) - p\left(S^{-1}\left[u^{\alpha} S\left[L\left(\sum_{n=0}^{\infty} p^n \underline{w}_n(x,t)) + \sum_{n=0}^{\infty} p^n A_n - g(x,t) \right)\right]\right]\right)$$

Equating the terms with identical powers of p, it can obtain a series of equations as follows:

$$p^0 : \underline{w}_0(x,t) = G(x,t)$$

$$p^1 : \underline{w}_1(x,t) = -S^{-1}[u^{\alpha} S[L\underline{w}_0(x,t) + A_0 - g(x,t)]]$$

$$p^2 : \underline{w}_2(x,t) = -S^{-1}[u^{\alpha} S[L\underline{w}_1(x,t) + A_1 - g(x,t)]]$$

and similarly

$$p^n : \underline{w}_n(x,t) = -S^{-1}[u^{\alpha} S[L\underline{w}_{n-1}(x,t) + A_{n-1} - g(x,t)]]$$

Likewise, by doing the same calculation equation (3) will be

$$\sum_{n=0}^{\infty} p^n \overline{w}_n(x,t) = G(x,t) - p\left(S^{-1}\left[u^{\alpha} S\left[L\left(\sum_{n=0}^{\infty} p^n \overline{w}_n(x,t)) + \sum_{n=0}^{\infty} p^n A_n - g(x,t) \right)\right]\right]\right)$$

Equating the terms with identical powers of p, it can obtain a series of equations as follows:

$$p^0 : \overline{w}_0(x,t) = G(x,t)$$

$$p^1 : \overline{w}_1(x,t) = -S^{-1}[u^{\alpha} S[L\overline{w}_0(x,t) + A_0 - g(x,t)]]$$

$$p^2 : \overline{w}_2(x,t) = -S^{-1}[u^{\alpha} S[L\overline{w}_1(x,t) + A_1 - g(x,t)]]$$

and similarly

$$p^n : \overline{w}_n(x,t) = -S^{-1}[u^{\alpha} S[L\overline{w}_{n-1}(x,t) + A_{n-1} - g(x,t)]]$$

proceeding in the same manner, the rest of the components $(\underline{w}_n(x,t), \overline{w}_n(x,t))$ can be completely found and the series solutions is thus entirely determined. Then, approximate the analytical solution $\tilde{w}(x,t) = [\underline{w}(x,t), \overline{w}(x,t)]$ by truncated series as:

$$\underline{w}(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} p^n \underline{w}_n(x,t)$$

$$\overline{w}(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} p^n \overline{w}_n(x,t)$$

4. NUMERICAL EXAMPLES

In this section, the solution of the fuzzy fractional heat equation is determined.

Example 4.1

Consider the following fuzzy fractional heat equation

$$\frac{\partial^{\alpha} \tilde{w}}{\partial t^{\alpha}} = \frac{x^2}{2} \frac{\partial^2 \tilde{w}}{\partial t^2}, \quad 0 < x < 1, \quad t > 0, \quad 0 < \alpha < 1 \tag{4.1}$$

with initial condition $\tilde{w}(x,0) = \tilde{f}(x) = \tilde{k}x^2$, $0 < x < 1$, where

$$\tilde{k} = [\underline{k}(r), \overline{k}(r)] = [0.75 + 0.25r, 1.25 - 0.25r]$$

The exact solution of (4.1) is,

$$\tilde{w}(x,t) = \tilde{k}\left(\frac{x^2}{2}\right)\sum_{n=0}^{\infty}\frac{(t^{\alpha})^n}{\Gamma(n\alpha+1)}$$

The parametric form of (4.1) is

$$\frac{\partial^{\alpha}\underline{w}}{\partial t^{\alpha}} = \frac{x^2}{2}\frac{\partial^2\underline{w}}{\partial x^2} \tag{4.2}$$

$$\frac{\partial^{\alpha}\overline{w}}{\partial t^{\alpha}} = \frac{x^2}{2}\frac{\partial^2\overline{w}}{\partial x^2} \tag{4.3}$$

Taking the Sumudu transform on both sides of equation (4.2),

$$S[D_t^{\alpha}\underline{w}(x,t)] = \frac{x^2}{2}S[D_x^2\underline{w}(x,t)], \quad 0 < x < 1, \quad t > 0$$

Operating with the Sumudu inverse on both sides

$$\underline{w}(x,t) = (0.75 + 0.25r)x^2 + S^{-1}\left[u^{\alpha}S\left[\frac{x^2}{2}D_x^2\underline{w}(x,t)\right]\right]$$

By applying the homotopy perturbation method,

$$\sum_{n=0}^{\infty}p^n\underline{w}_n(x,t) = (0.75 + 0.25r)x^2 + pS^{-1}\left[u^{\alpha}S\left[\frac{x^2}{2}D_x^2\sum_{n=0}^{\infty}p^n\underline{w}_n(x,t)\right]\right]$$

Equating the terms with identical powers of *p*,

$$p^0 : w_0(x,t) = (0.75 + 0.25r)x^2$$

$$p^1 : \underline{w}_1(x,t) = (0.75 + 0.25r)x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$p^2 : \underline{w}_2(x,t) = (0.75 + 0.25r)x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$p^3 : \underline{w}_3(x,t) = (0.75 + 0.25r)x^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$p^n : \underline{w}_n(x,t) = (0.75 + 0.25r)x^2\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

Likewise, by doing the same calculation (4.3) will be

$$\sum_{n=0}^{\infty}p^n\overline{w}_n(x,t) = (1.25 - 0.25r)x^2 + pS^{-1}\left[u^{\alpha}S\left[\frac{x^2}{2}D_x^2\sum_{n=0}^{\infty}p^n\overline{u}_n(x,t))\right]\right]$$

Equating the terms with identical powers of p,

$$p^0 : \overline{w}_0(x,t) = (1.25 - 0.25r)x^2$$

$$p^1 : \overline{w}_1(x,t) = (1.25 - 0.25r)x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$p^2 : \overline{w}_2(x,t) = (1.25 - 0.25r)x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$p^3 : \overline{w}_3(x,t) = (1.25 - 0.25r)x^2 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$p^n : \overline{w}_n(x,t) = (1.25 - 0.25r)x^2 \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$\vdots$$

Then,

$$\underline{w}(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} p^n \underline{w}_n(x,t) = (0.75 + 0.25r)x^2 \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

Similarly,

$$\overline{w}(x,t) = \lim_{N\to\infty} \sum_{n=0}^{\infty} p^n \overline{w}_n(x,t) = (1.25 - 0.25r)x^2 \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

The approximate fuzzy solution of the method (HPSTM) of order 10 is plotted for $\alpha =$ 0.5, 0.75, 0.95, 1 and $x =$ 0.5.

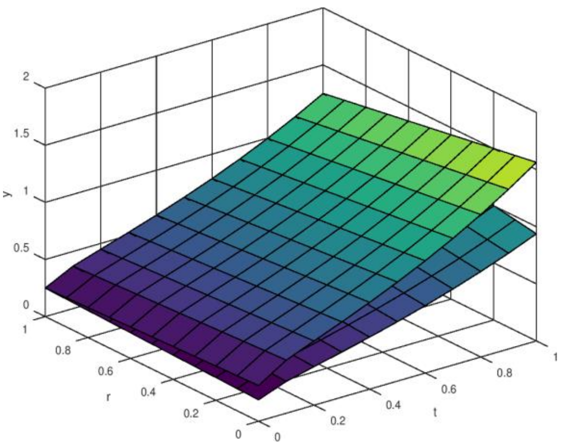

Fig. 4.1: Approximate Solution for Example 4.1 at α = 0.5 and x = 0.5

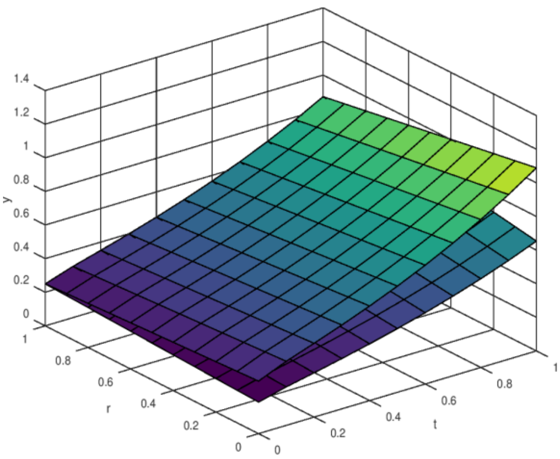

Fig. 4.2: Approximate Solution for Example 4.1 at α = 0.75 and x = 0.5

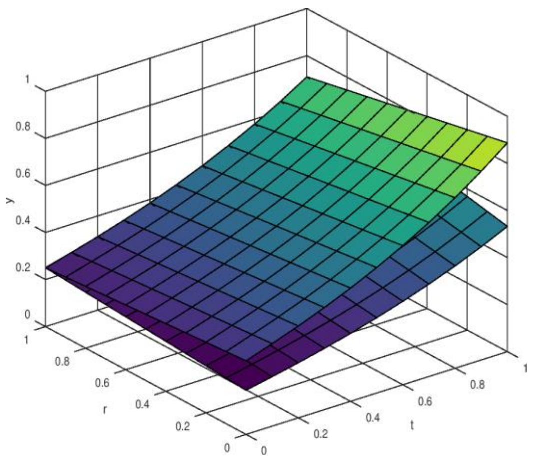

Fig. 4.3: Approximate Solution for Example 4.1 at α = 0.95 and x = 0.5

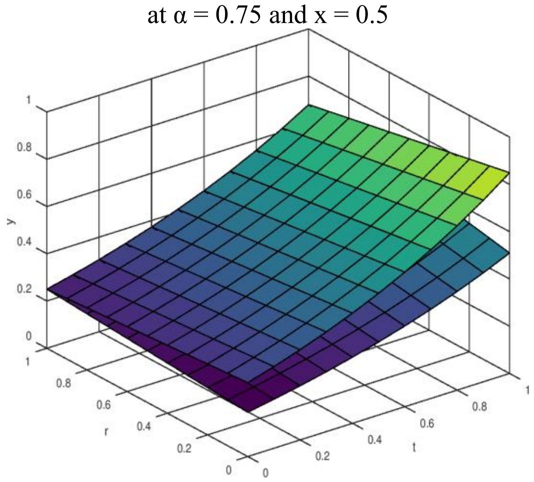

Fig. 4.4: Approximate Solution for Example 4.1 at α = 1 and x = 0.5

r	α=0.7		α=0.8		α=1	
	$\underline{w}(t,r)$	$\overline{w}(t,r)$	$\underline{w}(t,r)$	$\overline{w}(t,r)$	$\underline{w}(t,r)$	$\overline{w}(t,r)$
0	4.89E-08	8.15E-08	2.00E-09	3.33E-09	2.39E-12	3.99E-12
0.1	5.06E-08	7.99E-08	2.07E-09	3.27E-09	2.47E-12	3.91E-12
0.2	5.22E-08	7.83E-08	2.13E-09	3.20E-09	2.55E-12	3.83E-12
0.3	5.38E-08	7.67E-08	2.20E-09	3.13E-09	2.63E-12	3.75E-12
0.4	5.55E-08	7.50E-08	2.27E-09	3.07E-09	2.71E-12	3.67E-12
0.5	5.71E-08	7.34E-08	2.33E-09	3.00E-09	2.79E-12	3.59E-12
0.6	5.87E-08	7.18E-08	2.40E-09	2.93E-09	2.87E-12	3.51E-12
0.7	6.03E-08	7.01E-08	2.47E-09	2.87E-09	2.95E-12	3.43E-12
0.8	6.20E-08	6.85E-08	2.53E-09	2.80E-09	3.03E-12	3.35E-12
0.9	6.36E-08	6.69E-08	2.60E-09	2.73E-09	3.11E-12	3.27E-12
1	6.52E-08	6.52E-08	2.67E-09	2.67E-09	3.19E-12	3.19E-12

Table 4.1: Error between Exact & Approximate solution (HPSTM) for Example 4.1 at t=0.5 & x=0.5.

From figure 4.1-4.4, the surface plots are given for the solution of Example 4.1 corresponding to different fractional order (α = 0.5, 0.75, 0.95 and 1) for various values of both t and r (varies from 0 to 1) at x=0.5.

Example 4.2

Consider the following fuzzy fractional heat equation

$$\frac{\partial^\alpha \tilde{w}}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 \tilde{w}}{\partial t^2}, \qquad 0 < x < 1, \qquad t > 0, \qquad 0 < \alpha < 1 \tag{4.4}$$

with initial condition $\tilde{w}(x,0) = \tilde{f}(x) = \tilde{k}x^2$, $0 < x < 1$, where $\tilde{k} = [\underline{k}(r), \overline{k}(r)] = [r-1, 1-r]$.

The exact solution of (4.4) is,

$$\tilde{w}(x,t) = \tilde{k}\left(\frac{x^2}{2}\right)\sum_{n=0}^{\infty}\frac{(t^\alpha)^n}{\Gamma(n\alpha+1)}$$

The parametric form of (4.4) is

$$\frac{\partial^\alpha \underline{w}}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 \underline{w}}{\partial x^2} \tag{4.5}$$

$$\frac{\partial^\alpha \overline{w}}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 \overline{w}}{\partial x^2} \tag{4.6}$$

Taking the Sumudu transform on both sides of equation (4.5),

$$S[D_t^\alpha \underline{w}(x,t)] = \frac{x^2}{2} S[D_x^2 \underline{w}(x,t)], \qquad 0 < x < 1, \qquad t > 0$$

and

$$u^{-\alpha} S[\underline{w}(x,t)] - u^{-\alpha}\underline{w}(x,0) = S\left[\frac{x^2}{2} D_x^2 \underline{w}(x,t)\right]$$

Using the property of Sumudu transform and the initial condition in (4.4),

$$S[\underline{w}(x,t)] = (r-1)x^2 + u^\alpha S[D_x^2 \underline{w}(x,t)]$$

Operating with the Sumudu inverse on both sides

$$\underline{w}(x,t)=(r-1)x^2+S^{-1}\left[u^{\alpha}S\left[\frac{x^2}{2}D_x^2\underline{w}(x,t)\right]\right]$$

By applying the homotopy perturbation method,

$$\sum_{n=0}^{\infty}p^n\underline{w}_n(x,t)=(r-1)x^2+pS^{-1}\left[u^{\alpha}S\left[\frac{x^2}{2}D_x^2\sum_{n=0}^{\infty}p^n\underline{w}_n(x,t))\right]\right]$$

Equating the terms with identical powers of p,

$$p^0:\underline{w}_0(x,t)=(r-1)x^2$$

$$p^1:\underline{w}_1(x,t)=(r-1)x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$p^2:\underline{w}_2(x,t)=(r-1)x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$p^3:\underline{w}_3(x,t)=(r-1)x^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$p^n:\underline{w}_n(x,t)=(r-1)x^2\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

Likewise, by doing the same calculation (4.6) will be

$$\sum_{n=0}^{\infty}p^n\overline{w}_n(x,t)=(1-r)x^2+pS^{-1}\left[u^{\alpha}S\left[\frac{x^2}{2}D_x^2\sum_{n=0}^{\infty}p^n\overline{w}_n(x,t))\right]\right]$$

Equating the terms with identical powers of p,

$$p^0:\overline{w}_0(x,t)=(1-r)x^2$$

$$p^1:\overline{w}_1(x,t)=(1-r)x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$p^2:\overline{w}_2(x,t)=(1-r)x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$p^3:\overline{w}_3(x,t)=(1-r)x^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$p^n:\overline{w}_n(x,t)=(1-r)x^2\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$\vdots$$

Then,

$$\underline{w}(x,t)=\lim_{N\to\infty}\sum_{n=0}^{N}p^n\underline{w}_n(x,t)=(r-1)x^2\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$\overline{u}(x,t)=\lim_{N\to\infty}\sum_{n=0}^{\infty}p^n\overline{u}_n(x,t)=(1-r)x^2\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

The approximate fuzzy solution of the method (HPSTM) of order 10 is plotted for α=0.5, 0.75, 0.95, 1 and x =0.5.

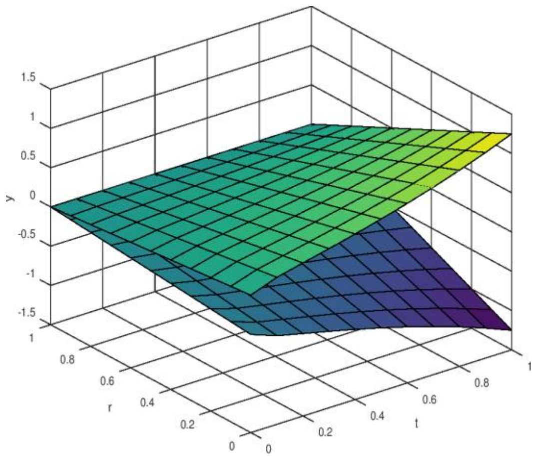

Fig.4.5: Approximate Solution for Example 4.2 at α = 0.5 and x = 0.5

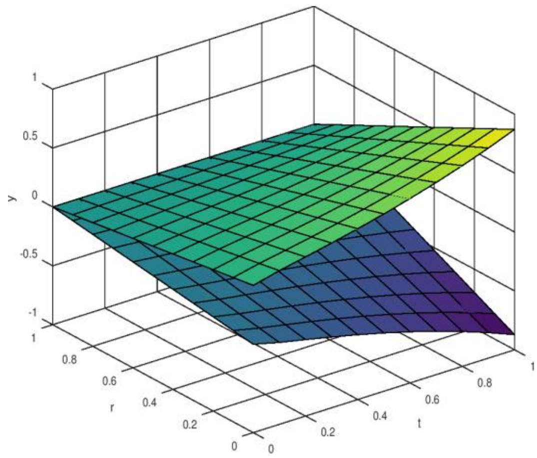

Fig.4.6: Approximate Solution for Example 4.2 at α = 0.75 and x = 0.5

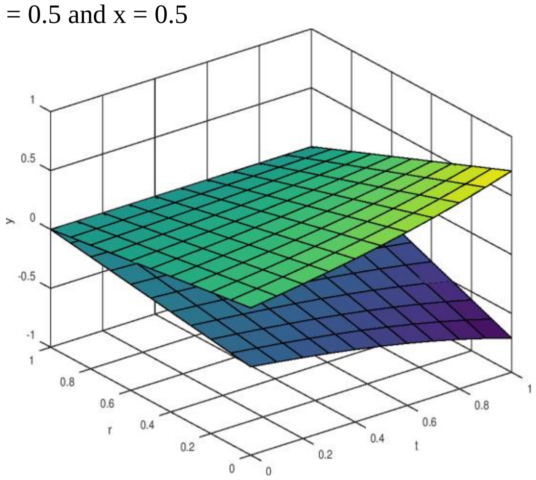

Fig.4.7: Approximate Solution for Example 4.2 at α = 0.95 and x = 0.5

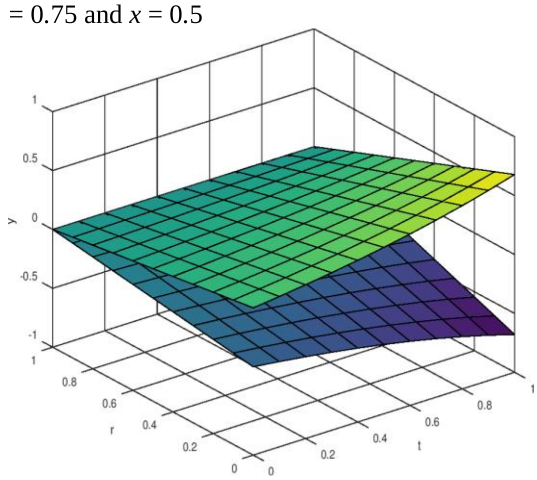

Fig.4.8: Approximate Solution for Example 4.2 at α = 1 and x = 0.5

r	α=0.7		α=0.8		α=1	
	$\underline{w}(t,r)$	$\overline{w}(t,r)$	$\underline{w}(t,r)$	$\overline{w}(t,r)$	$\underline{w}(t,r)$	$\overline{w}(t,r)$
0	6.52E-08	6.52E-08	2.67E-09	2.67E-09	3.19E-12	3.19E-12
0.1	5.87E-08	5.87E-08	2.40E-09	2.40E-09	2.87E-12	2.87E-12
0.2	5.22E-08	5.22E-08	2.13E-09	2.13E-09	2.55E-12	2.55E-12
0.3	4.57E-08	4.57E-08	1.87E-09	1.87E-09	2.23E-12	2.23E-12
0.4	3.91E-08	3.91E-08	1.60E-09	1.60E-09	1.91E-12	1.91E-12
0.5	3.26E-08	3.26E-08	1.33E-09	1.33E-09	1.60E-12	1.60E-12
0.6	2.61E-08	2.61E-08	1.07E-09	1.07E-09	1.28E-12	1.28E-12
0.7	1.96E-08	1.96E-08	8.00E-10	8.00E-10	9.57E-13	9.57E-13
0.8	1.30E-08	1.30E-08	5.33E-10	5.33E-10	6.38E-13	6.38E-13
0.9	6.52E-09	6.52E-09	2.67E-10	2.67E-10	3.19E-13	3.19E-13
1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00

Table 4.2: Error between Exact & Approximate solution (HPSTM) for Example 4.2 at t=0.5 & x=0.5

From Fig. 4.5-4.8, the surface plots are given for the solution of Example 4.2 corresponding to different fractional order (α = 0.5, 0.75, 0.95, and 1) for various values of both t and r (varies from 0 to 1) at $x = 0.5$.

5. CONCLUSION

In this study, the HPSTM is successfully used for fuzzy fractional heat equations. The benefits of using this technique are: This approach can be used to solve fuzzy fractional ordinary and partial differential equations that are both linear and nonlinear. It employs a straightforward method to evaluate the solution for FFBPM. The given figures demonstrate that it is possible to calculate the exact solution for different values of the fractional orders δ and t as well as r. It may be inferred that the suggested approach is rapid, exact, and simple to apply and produce excellent outcomes. As fractional order differential systems in uncertain environments, this method has been used in the future for many significant applications in engineering and science.

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