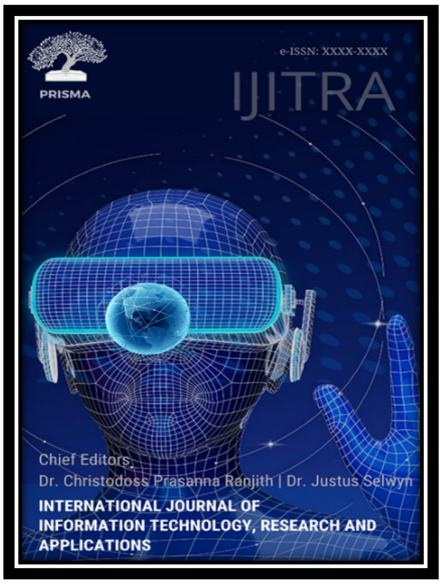

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Domination Polynomials of Unidirectional Star Graphs

Domination Polynomials of Unidirectional Star Graphs

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ABSTRACT

Let G be a finite directed graph without isolated node. The domination polynomial of G is defined by $D(G,x)=\sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$ where $d(G,i)$ is the cardinality of the set of dominating sets of G with i nodes. In this paper, we study the domination polynomial of unidirectional star graphs.
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I. INTRODUCTION

A directed graph (digraph) is a pair $G = (V, A)$, where V is a set of nodes and $A \subseteq V \times V$ is a set of arcs, i.e., ordered pairs of nodes. We say that node u dominates node v (or node v is dominated by node u) if arc (u, v) is in G. Arc (u, v) is denoted as $u \rightarrow v$ graphically. In digraph G, the in-degree of node v is the number of arcs directed into v. The out-degree of node u is the number of arcs going out of u.

II. IN DOMINATING SETS AND OUT-DOMINATING SETS

In digraph $G = (V, A)$, a set of nodes $D \subseteq V$ is a dominating set of G if each node $v \in V - D$ is dominated by at least a node in D. A minimal dominating set D_m is a dominating set with no proper subset of D_m as a dominating set. A minimum dominating set D_m is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the domination number of G and is denoted by $\gamma(G)$.
Let $G = (V, E)$ be a directed graph(digraph) with the set of nodes $V(G)$ and the set of arcs $E(G)$ such that each arc $(u, v) \epsilon E(G)$ is directed and u is said to be a predecessor of v, v is a successor of u, and u dominates v. Assume that G contains no loops or multiple arcs. For any node $v \in V$, the open in-neighborhood of v is the set $N^-(v) = \{u \in V: (u, v) \in E\}$ and the closed in-neighborhood of v is the set $N^-[v] = N^-(v) \cup \{v\}$. For a subset $S \subseteq V$, the open in-neighborhood of S is $N^-(S) = \bigcup_{v \in S} N^-(v)$, and the closed in-neighborhood of S is $N^-[S] = \bigcup_{v \in S} N^-[v]$. Analogously, for any node v, we define the open out-neighborhood of v is the set $N^+(v) = \{u \in V: (v, u) \in E\}$ and the closed out-neighborhood of v is the set $N^+[v] = N^+(v) \cup \{v\}$. For a set $S \subseteq V$, the open out-neighborhood of S is $N^+(S) = \bigcup_{v \in S} N^+(v)$and the closed out-neighborhood of S is $N^+[v] = N^+(v) \cup \{v\}$.

A subset $S \subseteq V$ is an in-dominating set of G, if $N^-[S] = V$, that is, every node in V is dominated by least one node in S. The in-domination number $\gamma^-(G)$ is the minimum cardinality of an in-dominating set in G. A subset $S \subseteq V$ is a out-dominating set of G, if $N^+[S] = V$, or equivalently, every node in V is dominated by at least one node in S. The out-domination number $\gamma^+(G)$ is the minimum cardinality of an out-dominating set in G.

III. IN DOMINATION POLYNOMIAL AND OUT DOMINATION POLYNOMIAL

Definition 3.1

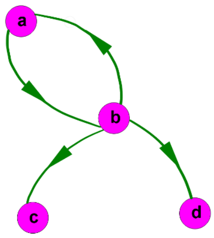
Let $d^-(G,i)$ be the cardinality of the in-dominating sets of size i, and let $d^+(G,i)$ be the cardinality of the out-dominating sets of size i of a digraph G. Then the in-domination polynomial $D^-(G,x)$ and the out-domination polynomial $D^+(G,x)$ of G are defined as follows:

$$D^-(G,x) = \sum_{i=\gamma^-(G)}^{n} d^-(G,i)x^i,$$

$$D^+(G,x) = \sum_{i=\gamma^+(G)}^{n} d^+(G,i)x^i.$$

Example 3.2

Consider the below digraph.



The only out-dominating set of cardinality 1 is $\{b\}$. So, $d^+(G,1) = 1$. Since out-dominating sets of cardinality 2 are $\{a,b\}, \{b,c\}$, and $\{b,d\}$, $d^+(G,2) = 3$. The out-dominating sets of cardinality 3 are $\{a,b,c\}, \{a,b,d\}$, and $\{b,c,d\}$. So $d^+(G,3) = 3$. The only out-dominating set of cardinality 4 is $\{a,b,c,d\}$. $\therefore$ $d^+(G,4) = 1$.

Thus, the out-domination polynomial $D^+(G,x)$ is given by

$$D^+(G,x) = x + 3x^2 + 3x^3 + x^4$$

There are no in-dominating sets of cardinalities 1 and 2. The in-dominating sets of cardinality 3 are $\{a,c,d\}$, and $\{b,c,d\}$. $\therefore$ $d^-(G,3) = 2$. The only in-dominating set of cardinality 4 is $\{a,b,c,d\}$. $\therefore$ $d^-(G,4) = 1$.

Thus, the in-domination polynomial $D^-(G,x)$ is given by

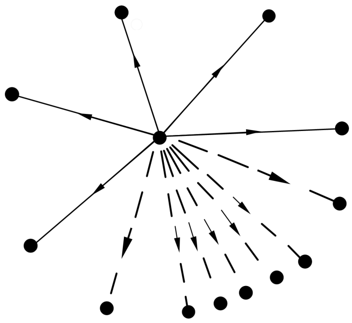
$$D^-(G,x) = 2x^3 + x^4$$

IV. IN-DOMINATION POLYNOMIALS AND OUT-DOMINATION POLYNOMIALS OF UNIDIRECTIONAL STAR

Lemma 4.1

The out-domination and in-dominations polynomials of unidirectional star graph with $n+1$ nodes and n arcs, n nodes of in-degree 1 and out-degree 0 are $x(1+x)^n$ and $x^n(1+x)$ respectively.

Proof:



Let S_n^+ denotes the directed graph with $n+1$ nodes and n arcs, n nodes of in-degree 1 and out-degree 0 . The singleton set containing the nucleus node out-dominates all other nodes of $S_n^+$and no other set of cardinality 1 does the same. Therefore, the number of out-dominating sets of cardinality 1 is 1. Any set of cardinality 2 out-dominates S_n^+ if and only if one of the nodes is the nucleus node. The number of such sets is n. Therefore, the number of out-dominating sets of cardinality 2 is $n = n\mathrm{C}_1$. Similarly, any set of cardinality 3 out-dominates S_n^+ if and only if one of the nodes is the nucleus node. The number of such sets is $n\mathrm{C}_2$. Therefore, the number of out-dominating sets of cardinality 3 is $n\mathrm{C}_2$. Thus we can conclude that the number of out-dominating sets of cardinality r is $n\mathrm{C}_{r-1}$. Clearly the number of out-dominating sets of cardinality $n+1$ is 1.

Hence the out-domination polynomial of S_n^+ is given by,

$$D^+(S_n^+, x) = x + n\mathrm{C}_1x^2 + n\mathrm{C}_2x^3 + \cdots + n\mathrm{C}_nx^{n+1}$$

$$= x(1 + n\mathrm{C}_1x + n\mathrm{C}_2x^2 + \cdots + n\mathrm{C}_nx^n)$$

$$= x(1+x)^n$$

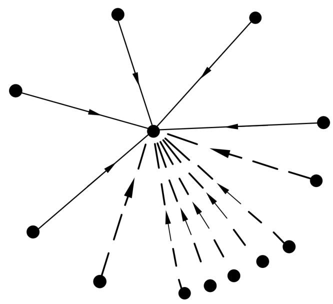
Any set of cardinality less than n does not in-dominate S_n^+. The set of all n non-nucleus nodes clearly in-dominates S_n^+. Therefore, the number of in-dominating sets of cardinality n is 1. Moreover, the set of all nodes of S_n^+ trivially in-dominates S_n^+ and the number of in-dominating sets of cardinality $n+1$ is 1. Hence the in-domination polynomial of S_n^+ is given by,

$$D^-(S_n^+, x) = x^n + x^{n+1} = x^n(1+x).$$

Lemma 4.2

The out-domination and in-dominations polynomials of unidirectional star graph with $n+1$ nodes and n arcs, n nodes of out-degree 1 and in-degree 0 are $x^n(1+x)$ and $x(1+x)^n$ respectively.

Proof:



Let S_n^- denotes the directed graph with $n+1$ nodes and n arcs, n nodes of out-degree 1 and in-degree 0. Any set of cardinality less than n does not out-dominate S_n^-. The set of all n non-nucleus nodes clearly out-dominates S_n^-. Therefore, the number of out-dominating sets of cardinality n is 1. Moreover, the set of all nodes of S_n^- trivially out-dominates S_n^- and the number of out-dominating sets of cardinality $n+1$ is 1.

Hence the out-domination polynomial of S_n^- is given by,

$$D^+(S_n^-, x) = x^n + x^{n+1} = x^n(1+x)$$

The singleton set containing the nucleus node in-dominates all other nodes of $S_n^-$and no other set of cardinality 1 does the same. Therefore, the number of in-dominating sets of cardinality 1 is 1.

Any set of cardinality 2 in-dominates S_n^- if and only if one of the nodes is the nucleus node. The number of such sets is n. Therefore, the number of in-dominating sets of cardinality 2 is $n = n\mathrm{C}_1$. Similarly, any set of cardinality 3 in-dominates S_n^- if and only if one of the nodes is the nucleus node. The number of such sets is $n\mathrm{C}_2$. Therefore, the number of in-dominating sets of cardinality 3 is $n\mathrm{C}_2$. Thus, we can conclude that the number of in-dominating sets of cardinality r is $n\mathrm{C}_{r-1}$. Clearly the number of in-dominating sets of cardinality $n+1$ is 1.

Hence the in-domination polynomial of S_n^- is given by,

$$D^-(S_n^-, x) = x + n\mathrm{C}_1x^2 + n\mathrm{C}_2x^3 + \cdots + n\mathrm{C}_nx^{n+1}$$

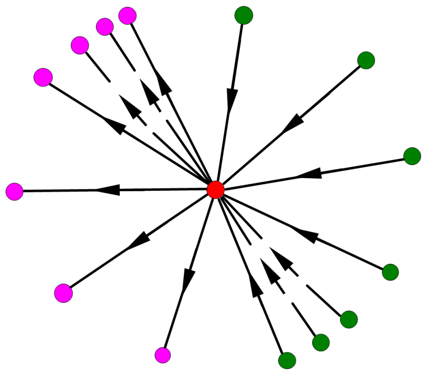
$$= x(1 + n\mathrm{C}_1x + n\mathrm{C}_2x^2 + \cdots + n\mathrm{C}_nx^n)$$

$$= x(1+x)^n$$

Remark 4.3

1. The out-domination polynomial of S_n^+ and in-domination polynomial of S_n^- are the same.
2. The out-domination polynomial of S_n^+ and in-domination polynomial of S_n^- are the same.

Theorem 4.4
The out-domination polynomial and the in-domination polynomial of a directed star graph with $n+m+1$ nodes, $n+m$ arcs, n nodes of out-degree 1 and in-degree 0, and m nodes of in-degree 1 and out-degree 0 are $x^{n+1}(1+x)^m$ and $x^{m+1}(1+x)^n$ respectively.
Proof:



Let $S_{n,m}$ denotes the directed graph with $n+m+1$ nodes, $n+m$ arcs, n nodes of out-degree 1 and in-degree 0, and m nodes of in-degree 1 and out-degree 0.
The set of n nodes of out-degree 1 together with the nucleus node out-dominates $S_{n,m}$. Any set of nodes with a lesser cardinality does not out-dominate $S_{n,m}$ and such an out-dominating set is unique and its cardinality is $n+1$. The above mentioned set along with any one of the remaining m nodes clearly out-dominates $S_{n,m}$ and hence there are $m\mathrm{C}_1$ different sets of cardinality $n+2$. By a similar argument, there are $m\mathrm{C}_2$ different out-dominating sets of cardinality $n+3$ and so on. Finally, there is only one out-dominating set of cardinality $n+m+1$.
Thus the out-domination polynomial of $S_{n,m}$ is given by,

$$\begin{aligned} D^+(S_{n,m},x) &= x^{n+1} + m\mathrm{C}_1 x^{n+2} + m\mathrm{C}_2 x^{n+3} + \cdots + m\mathrm{C}_m x^{n+m+1} \\ &= x^{n+1}(1 + m\mathrm{C}_1 x + m\mathrm{C}_2 x^2 + \cdots + m\mathrm{C}_m x^m) \\ &= x^{n+1}(1+x)^m \end{aligned}$$

The set of m nodes of in-degree 1 together with the nucleus node in-dominates $S_{n,m}$. Any set of nodes with a lesser cardinality does not in-dominate $S_{n,m}$ and such an in-dominating set is unique and its cardinality is $m+1$. The above mentioned set along with any one of the remaining n nodes clearly in-dominates $S_{n,m}$ and hence there are $n\mathrm{C}_1 = n$ different sets of cardinality $m+2$. By a similar argument, there are $n\mathrm{C}_2$ different in-dominating sets of cardinality $m+3$ and so on. Finally, there are $n\mathrm{C}_n = 1$ in-dominating set of cardinality $m+n+1$.

Thus the in-domination polynomial of $S_{n,m}$ is given by,

$$\begin{aligned} D^-(S_{n,m},x) &= x^{m+1} + n\mathrm{C}_1 x^{m+2} + n\mathrm{C}_2 x^{m+3} + \cdots + n\mathrm{C}_n x^{m+n+1} \\ &= x^{m+1}(1 + n\mathrm{C}_1 x + n\mathrm{C}_2 x^2 + \cdots + n\mathrm{C}_n x^n) \\ &= x^{m+1}(1+x)^n \end{aligned}$$

V. CONCLUSION

In this paper, the authors have derived the domination polynomials of directed star graphs. Presently, the authors are working on the same concept with the different operations of star graphs.

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